6.2.3 Since $\psi(\theta) = \theta^2$ is a 1-1 transformation of θ when θ is restricted to $[0, 1]$, we can apply Theorem 6.2.1, so the MLE is $\psi(\hat{\theta}(x_1, ..., x_n)) = \bar{x}^2$.

6.2.5 The likelihood function is given by $L(\theta \mid x_1, ..., x_n) = \theta^{n\alpha_0} \exp(-n\bar{x}\theta)$, the log-likelihood function is given by $l(\theta \mid x_1, ..., x_n) = n\alpha_0 \ln \theta - n\bar{x}\theta$, and the score function is given by $S(\theta \mid x_1, ..., x_n) = n\alpha_0/\theta - n\bar{x}$. Solving the score equation gives $\hat{\theta}(x_1, ..., x_n) = \alpha_0/\bar{x}$. Note that since $\bar{x} > 0$ we have that

$$\left.\frac{\partial S(\theta \mid x_1, ..., x_n)}{\partial \theta}\right|_{\theta=\frac{\alpha_0}{\bar{x}}} = -\left.\frac{n\alpha_0}{\theta^2}\right|_{\theta=\frac{\alpha_0}{\bar{x}}} = -\frac{n\bar{x}^2}{\alpha_0} < 0,$$

so $\hat{\theta} = \alpha_0/\bar{x}$ is the MLE.

6.2.8 The likelihood function is given by

$$L(\beta \mid x_1, ..., x_n) = \beta^n \left(\prod_{i=1}^n x_i\right)^{\beta-1} \exp\left(-\sum_{i=1}^n x_i^\beta\right),$$

the log-likelihood function is given by

$$l(\beta \mid x_1, ..., x_n) = n \ln \beta + (\beta - 1)\left(\sum_{i-1}^n \ln x_i\right) - \sum_{i=1}^n x_i^\beta,$$

and the score equation is given by

$$S(\beta \mid x_1, ..., x_n) = \frac{n}{\beta} + \sum_{i=1}^n \ln x_i - \sum_{i=1}^n x_i^\beta \ln x_i = 0.$$

6.2.11 The parameter of the interest is changed to the volume $\eta = \mu^3$ from the length of a side μ. Then the likelihood function is also changed to

$$L_v(\eta|s) = L_v(\mu^3|s) = L_l(\mu|s)$$

where L_v is the likelihood function when the volume parameter $\eta = \mu^3$ is of the interest and L_l is the likelihood function of the length of a side parameter μ. The maximizer η of $L_v(\eta|s)$ is also a maximizer of $L_l(\eta^{1/3}|s)$. In other words, the MLE is invariant under 1-1 smooth parameter transformations. Hence, the MLE of η is equal to $\hat{\mu}^3 = (3.2\text{cm})^3 = 32.768\text{cm}^3$.

6.2.12 The likelihood function is given by

$$L(\sigma^2|x_1,\ldots,x_n) = (\sigma^2)^{-n/2}\exp(-\sum_{i=1}^n (x_i-\mu_0)^2/(2\sigma^2)).$$

The derivative of the log-likelihood function with respect to σ^2 is

$$-\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^n (x_i-\mu_0)^2.$$

Hence, the maximum likelihood estimator is $\hat{\sigma}^2 = n^{-1}\sum_{i=1}^n (x_i-\mu_0)^2$. If the location parameter μ_0 is also unknown, then the estimator for σ^2 is $\tilde{\sigma}^2 = (n-1)^2\sum_{i=1}^n (x_i-\bar{x})^2$ as in Example 6.2.6. The difference of two estimators is

$$\begin{aligned}\hat{\sigma}^2 - \tilde{\sigma}^2 &= \frac{1}{n}\sum_{i=1}^n (x_i-\mu_0)^2 - \frac{1}{n-1}\sum_{i=1}^n (x_i-\bar{x})^2 \\ &= -\frac{1}{n(n-1)}\sum_{i=1}^n (x_i-\bar{x})^2 + (\bar{x}-\mu_0)^2 \\ &= -s^2/n + (\bar{x}-\mu_0)^2.\end{aligned}$$

In the second equality, the expansion $(x_i-\mu_0)^2 = (x_i-\bar{x})^2 + (\bar{x}-\mu_0)^2 + 2(\bar{x}-\mu_0)(x_i-\bar{x})$ is used. Thus, the summation becomes $\sum_{i=1}^n (x_i-\mu_0)^2 = \sum_{i=1}^n (x_i-\bar{x})^2 + n(\bar{x}-\mu_0)^2 + 2(\bar{x}-\mu_0)\sum_{i=1}^n (x_i-\bar{x})$. The last term is zero because the summation in the last term is zero. By the law of large numbers, $\bar{x} \xrightarrow{P} \mu_0$ and $s^2 \xrightarrow{P} \sigma^2$. Hence, the difference $\hat{\sigma}^2 - \tilde{\sigma}^2 \xrightarrow{P} 0$ as $n \to \infty$.